Baryon Isovector Electric Properties and the Large N_c and Chiral Limits

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Abstract

A model independent calculation is given for the nucleon isovector electric charge radius which is valid in the limit $N_c \to \infty$, $m_\pi \to 0$, $N_c m_\pi$ fixed. This expression reduces to that of the Skyrme model in the limit $N_c m_\pi \to \infty$.

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The study of baryons in the large N_c and chiral limits is an interesting and subtle problem. The interest stems largely from the belief that for many purposes $N_c = 3$ is large enough so that an expansion of physical quantities in a $1/N_c$ expansion gives at least a reasonable, if crude, description of hadronic properties while at the same time the up and down quark masses are light enough so that an expansion in terms of m_q is also reasonable. However, for some important physical quantities the limits $m_q \to 0$ and $N_c \to \infty$ do not commute [1,2]. That is, if such a quantity is treated as a function of m_q, then the $1/N_c$ expansion is not uniformly convergent. This implies that for such quantities expansions in $1/N_c$ and in m_q cannot be consistent with each other. This issue is significant for both theoretical and practical reasons. On the theoretical side it helps make manifest the nature of QCD in these important limits. As a practical matter, if the two expansions are inconsistent, at least one of the expansions must be invalid. Thus, one may learn about places where one expects the expansions to fail and, perhaps, one may be able to correct for the failure of one of the expansions.

This letter will explore the large N_c and chiral behavior of isovector electric properties of baryons. It has long been suspected that some of these properties are afflicted with problems associated with noncommuting limits. The evidence for this has been in the Skyrme model [3] in which Adkins and Nappi [1] found that the behavior of the isovector charge radius of the proton in the limit $m_\pi^2 \to 0$ goes as m_π^{-1} while in conventional chiral perturbation theory one finds the same quantity goes as $\text{Log}(m_\pi)$ times a coefficient proportional to N_c [4,5]. They speculated that this was due to the lack of commutativity of the two limits. Here it will be shown that that this speculation is correct and the underlying nature of this behavior will be explained in terms of the role played by the Δ isobar.

More specifically, it will be shown: (i) that the leading chiral behavior of the isovector charge radius is calculable in large N_c chiral perturbation theory—a generalization of usual chiral perturbation which is valid in the limit $m_\pi \to 0$, $N_c \to \infty$, $N_c m_\pi \to$ fixed. It is given by

$$\langle \text{N}|r^2|\text{N}\rangle_{I=1} = \frac{5\,g_A^2}{4\,\pi^2 f_\pi^2} \frac{\delta m}{\sqrt{m_\pi^2 - \delta m^2}} \tan^{-1}\left(\sqrt{\frac{m_\pi - \delta m}{m_\pi + \delta m}}\right) + C \tag{1}$$

where $\delta m = M_\Delta - M_N$ and C is a constant subleading in either $1/N_c$, m_π^2 or both. (ii) There is a model-independent prediction of the isovector charge radius for any Skyrme model or other large N_c hedgehog model, in terms of physical observables, which is valid in the limit $m_\pi \to 0$, $N_c \to \infty$ with the large N_c limit taken first. It is given by:

$$\langle \text{N}|r^2|\text{N}\rangle_{I=1} = \frac{5\,g_A^2\,\delta m}{16\,\pi\,f_\pi^2\,m_\pi} \tag{2}$$

(iii) This model-independent Skyrme result differs from that obtained in usual chiral perturbation theory with the limit $m_\pi \to 0$ taken first. (iv) However, the model-independent prediction from all Skyrme-type models is equivalent with the large N_c chiral perturbation theory prediction provided the large N_c limit is taken prior to the the chiral limit. This provides support for the conjecture that the Skyrme model correctly reproduces all model-independent predictions of large N_c QCD. (v)The $1/N_c$ expansion for the isovector charge

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radius may have a finite radius of convergence. There is significant evidence that $N_c = 3$ is outside this radius of convergence. (vi) Analogous results can be obtained for other isovector electromagnetic observables such as the higher moments of the electric form factor of the nucleon or the E2 N-Δ quadrupole transition.

The $\Delta - N$ mass splitting plays a central role in eqs. (1) and (2). This is not unexpected. As has been noted previously [2], the fact that

$$M_\Delta - M_N \sim 1/N_c \qquad (3)$$

can easily lead to noncommutativity of the large N_c and chiral limits. The reason for this clear. The leading nonanalytic behavior (in m_q or m_π^2) for the charge radius in chiral perturbation theory is due to small energy denominators coming from nucleon plus one pion states. However, in the $N_c \to \infty$ limit, $M_\Delta - M_N \to 0$ and there are new states with small energy denominators, such as Δ plus one pion states which can contribute to the nonanalytic behavior.

The Skyrme model [3] and other hedgehog soliton models such as the chiral-quark meson soliton model [6], the chiral or hybrid bag model [7] or the soliton approach to the Nambu–Jona-Lasinio (NJL) model [8] are based on both large N_c and approximate chiral symmetry. As all of these models behave identically for all of the issues discussed here, the phrase "Skyrme" model will be used in this letter to denote any of these hedgehog models. Approximate chiral symmetry is explicit in the form of the effective lagrangian. The large N_c nature is built into the method of calculation. These models are quantum field theories which in general are computationally intractable. However, in the large N_c limit, the models are calculable. The large N_c limit is essential in two ways. The first is that in this limit the problem reduces to a tractable classical field theory. Unfortunately, the classical solutions are hedgehog configurations which break symmetries of the theory; to obtain physical results one must project onto states with physical quantum numbers. There is a semiclassical projection method based on the separation of collective and intrinsic variables [9]. However, this technique is valid only in the large N_c limit. Thus, calculations in the Skyrme model implicitly correspond to working in the $N_c \to 0$ limit—i.e. to leading order in the $1/N_c$ expansion. The large N_c character of the model is also manifest in that the calculated baryon properties will be consistent with Witten's large N_c scaling rules for baryons [12].

It should be stressed that the semiclassical treatment of the problem is in an essential ingredient of the Skyrme-type models—unless one specifies a calculational scheme, these nonrenormalizable models are not well defined. The fact that the large N_c approximation is implicit in the semiclassical treatment used in all calculations implies that if some quantity has noncommuting large N_c and chiral limits, then one expects that semiclassical calculations in Skyrme-type models to correspond to taking the $N_c \to \infty$ first.

The value of the electric charge radius may be calculated in a straightforward way in Skyrme models using semiclassical techniques [9]. The result depends on the details of the model. However, it is easy to see that as $m_\pi \to 0$ the integrals are dominated by the contributions from the tail of the distribution. The pion fields in a hedgehog configuration may be written as $\pi_a(\vec{x}) = \pi(r)\hat{r}_a$ where a is the isospin direction. In any Skyrme-type model as $r \to \infty$, the pion field asymptotes to a p-wave Yukawa form with a field strength fixed by $g_{\pi NN}$:

$$\pi(r) \to \pi_{\text{asympt}}(r) = \frac{3g_{\pi NN}}{8\pi M_N}(m_\pi + 1/r)\frac{e^{-m_\pi r}}{r} \tag{4}$$

The isovector charge radius is given by

$$\langle r^2 \rangle_{I=1} = \frac{1}{\mathcal{I}} \int dr\, r^4 \frac{8\pi}{3}(\pi_{\text{asympt}}^2 + A(r)) \tag{5}$$

where $A(r)$ is a model-dependent function which goes to zero faster than π_{asympt}^2 and $\mathcal{I}$ is the moment of inertia. Inserting the asymptotic form into the integral, using the Goldberger-Trieman relation [13] $g_{\pi NN}f_\pi = g_A M_N$ (which is true to leading order in the pion mass) and the fact that $1/\mathcal{I} = 2/3(M_\Delta - M_N)$ (to leading order in the $1/N$ expansion) immediately yields eq. (2) + corrections which are higher order in either $1/N_c$ or m_q. It is worth stressing that this result is model independent in the sense that it applies to all Skyrme-type models.

Of course, this result is incompatible with the known chiral behavior of the charge radius in chiral perturbation theory [4,5]: $\langle r^2 \rangle_{I=1} = -\frac{5g_A^2+1}{8\pi^2 f_\pi^2}\text{Log}(m_\pi/\Lambda)$, where Λ is a mass parameter independent of m_q. This was originally deduced using dispersive methods [4]. In more modern language this result can be obtained from heavy baryon effective chiral lagrangian methods [14,15]. In such treatments the leading nonanalytic behavior in m_π^2 is believed to be completely reproduced by 1 pion loop feynman graphs. The term proportional to $5g_A^2$ comes from diagram a. in fig. () while the term proportional to unity comes from diagram b. To the extent, that we are interested in the large N_c behavior, the term proportional to unity may be dropped since $g_A \sim N_c$.

The nonanalytic behavior in m_π in diagram a. in this standard chiral perturbation theory analysis comes entirely from the nearly vanishing energy denominators associated with the π-N states. However, in the large N_c limit of QCD the Δ becomes nearly degenerate with the nucleon; the mass splitting goes as $1/N_c$. This can be seen directly in the Skyrme model [10,11] and can also be deduced in a model-independent way by large N_c consistency conditions [16]. Thus, in an effective lagrangian treatment of the combined large N_c and chiral limits, one must treat the Δ as an explicit low energy degree of freedom. Indeed, in the large N_c limit there is an entire band of low-lying states with $I = J$. As will be shown, however, for the leading nonanalytic behavior in $1/N_c$ and m_q of nucleon properties, only the N and Δ contribute.

Moreover, in the large N_c limit the π-N-Δ coupling is completely fixed. It is sufficient to specify the coupling in terms of a heavy baryon effective field theory (HBEFT) [14] in which the baryon degrees of freedom are treated nonrelativistically. The HBEFT is the natural formulation of chiral pertubation in which chiral power counting rules can be satisfied. The effective π-N-Δ coupling in the large N_c limit is given by

$$\mathcal{L}_{\pi N\Delta} = \frac{3g_A}{f_\pi}(\overline{\Delta}_{m',m_I'} X^{\Delta N\,ia}_{(m',m_I')(m,m_I)} N_{m,m_I}\, \partial_i \pi_a + \text{h.c.}) \tag{6}$$

where

$$X^{\Delta N\,ia}_{(m',m_I')(m,m_I)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1/2 & 1 & 3/2 \\ m_I & a & m_I' \end{pmatrix}\begin{pmatrix} 1/2 & 1 & 3/2 \\ m & i & m' \end{pmatrix} \tag{7}$$

The couplings in eqs. (6) and (7) are model independent; they can be obtained directly from the Skyrme model using conventional semiclassical techniques. This can also be obtained from large N_c consistency relations [16]. Indeed, the large N_c consistency relations can be used to show that any correction to these couplings is of relative order N_c^{-2}.

The notion that the Δ should be included as an explicit degree of freedom in chiral perturbation theory is not novel. It has been argued on phenomenolgical grounds [14] that the Δ (or more generally the decuplet in the three flavor case) should be included since it is light in the real ($N_c = 3$) world. The formal need to include the Δ explicitly in the combined large N_c and chiral limits is completely clear and has been seen unambiguously in the nonanalytic behavior of vector-isovector and scalar-isoscalar observables [2].

The leading order nonanalytic behavior in the combined large N_c and chiral limits (with $N_c m_\pi$ fixed) presumably comes from one pion loops with either nucleons or Deltas in the intermediate state. Thus, in addition to the graph in diagram (a.) of fig. () one needs to include diagram (c.). Note that other states in the $I = J$ band (*i.e.* states with $I = J \geq 5/2$) do not contribute to the leading nonanalytic behavior since, by isospin, at least two pions are required to connect to the $I = J = 5/2$ state and more to higher states while the leading nonanalytic behavior comes from a single pion loop. Evaluation of this loop is straightforward in HBEFT. The contribution to the isovector charge radius from this loop using the large N_c π-N-Δ vetex from eqs. (6) and (7) is

$$\langle |r^2|\rangle_{I=1}^{\Delta \text{loop}} \; = \; \frac{5g_A^2}{8\pi^2 f_\pi^2} \text{Log}(m_\pi/\Lambda) \; + \; \frac{5\,g_A^2}{4\,\pi^2 f_\pi^2} \frac{\delta m}{\sqrt{m_\pi^2 - \delta m^2}} \tan^{-1}\left(\sqrt{\frac{m_\pi - \delta m}{m_\pi + \delta m}}\right) \tag{8}$$

Note that the Log term precisely cancels the Log from diagram a. and the sum of the two immediately gives eq. (1).

It is useful to rewrite eq. (1) in terms $d \equiv \delta m/m_\pi$:

$$\langle \text{N}|r^2|\text{N}\rangle_{I=1} \; = \; \frac{5\,g_A^2 d}{16\pi f_\pi^2} \, S(d) + C \tag{9}$$

where C is subleading and $S(d)$ is a suppression factor associated with the fact that the Δ is not degenerate with the nucleon:

$$s(d) = \frac{4}{\pi} \frac{1}{\sqrt{1 - d^2}} \tan^{-1}\left(\sqrt{\frac{1 - d}{1 + d}}\right) \tag{10}$$

This suppression factor $s(d)$ is identical to the one that appears in the leading nonanalytic behavior of scalar-isoscalar and vector-isovector as shown in ref. [2].

A few observations are in order about eqs. (9) and (10). The first is that if one takes the large N_c limit with m_π held fixed then d goes to zero. Since $s(0) = 1$, it is easy to see that one immediately reproduces the Skyrme model result of eq. (2) in the formal $N_c \to \infty$ limit. This is significant in that it supports the conjecture that the Skyrme model correctly reproduces all of the model-independent predictions of large N_c QCD. A second issue is the treatment of eq. (10) when $d > 1$. This is of immediate consequence to the real world in which $d \sim 2$. The basic point is that the expression must be analytically continued. In

principle, there is an ambiguity in this continuation since there is a branch cut at $d = 1$. However, this ambiguity is trivially resolved by the realization that the imaginary part must vanish for this observable. Thus for $d > 1$

$$s(d) = \frac{4}{\pi} \frac{1}{\sqrt{d^2 - 1}} \tanh^{-1}\left(\sqrt{\frac{d-1}{d+1}}\right) \tag{11}$$

The formal limit of $m_\pi \to 0$ is easy to take. One obtains $\langle r^2 \rangle_{I=1} = -\frac{5g_A^2}{8\pi^2 f_\pi^2} \text{Log}(m_\pi/\delta m)$. This disagrees with the correct chiral perturbation theory result by an overall factor of $5g_A^2/(5g_A^2 + 1)$. The disagreement, however, is of order N_c^{-2} relative to the leading order and is due to the neglect of diagram (b.) of fig. ().

Using eqs. (9) and (11) and the physical values for g_A, δm, f_π and m_π yields $\langle r^2 \rangle_{I=1} = .72\,\text{fm}^2 + C$ where C is subleading. The experimental value is $.88\,\text{fm}^2$. Thus, this simple leading order result gives the correct result to within about 20% without including effects such as vector dominance.

It is worth stressing that the pure $1/N_c$ expansion (with m_π fixed) for any quantity which depends on $s(d)$ is highly problematic. The value of d in the real world is approximately 2.1 and $s(d_{\text{phys}}) \approx .47$. This is quite far from the large N_c value of unity. Ideally, one might hope that the disagreement between $s(0)$ and $s(d_{\text{phys}})$ can be explained in terms of higher order terms in the $1/N_c$ expansion which is just a power series in d around $d = 0$. Unfortunately the radius convergence of this series is unity. Thus, the physical value of $d \approx 2.1$ is beyond the radius of convergence. This implies that the inclusion of any finite number of subleading terms in the $1/N_c$ expansion will not improve the description and will, in general make things worse: denoting the Taylor expansion of $s(d)$ up to n^{th} order by $s^n(d)$ one finds, for example, $s^0(d_{\text{phys}}) = 1$, $s^1(d_{\text{phys}}) \approx -.34$, $s^2(d_{\text{phys}}) \approx 1.8$ and $s^3(d_{\text{phys}}) \approx -2.06$. The obvious lesson from this is that the pure $1/N_c$ is not likely to be convergent for the isovector charge radius and is certainly not convergent if it is dominated by pionic tail effects. This problem is particularly serious for Skyrme-type models—the only systematic calculational scheme currently known for these models is the pure $1/N_c$ expansion.

It is clear that analogous effects will follow for any isovector electric property, such as higher moments of the electric form factor or moments of the N-Δ transition form factor. The Skyrme model will correctly reproduce the leading order large N_c and chiral behavior provided that the large N_c limit is taken first. However, the corrections to this leading order $1/N_c$ behavior for physical parameters will be large and the $1/N_c$ expansion will not converge.

The distinction between the leading order large N_c results and the physical results is particularly important in the case of properties of the Δ or of the N-Δ transition. The difficulty is that for $N_c \to \infty$ the Δ is stable and quanitities such as the Δ charge radius or quadrupole moment have an unambiguous meaning in terms of measureable quantities. However, as N_c decreases to the point that the Δ becomes unstable these quanitities become ambiguous in the sense that there is no simple model-independent connection between the calculated quanitity and experimental observables.

An important example of this is the N-Δ electric quadrupole transition matrix element. In the large N_c and chiral limits this matrix element with the large N_c limit taken first, the matrix element corresponds to a well-defined observable directly proportional to the isovector

charge radius. A quasi-model independent relationship between the two [11] becomes exact in the chiral limit. This gives a transition quadrupole matrix element which is much larger than seen in typical quark models. However, as one increases d beyond unity the matrix element ceases to have any well-defined experimental signature. Thus, it is hard to use the quadrupole transition matrix element to distinguish between various models of the baryon.

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Fig. 1: Feynman graphs for leading chiral behavior of $\langle r^2 \rangle_{I=1}$. The cross represents an insertion of the 0th component of the electromagnetic current, the dotted line is a pion propagator, the solid line is a nucleon propagator and the double line is the Δ propagator.